Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE CONFORMAL MAPPING OF A REGION INTO A PART OF ITSELF.

BY J. F. RITT.

1. This note will extend to domains of any degree of connectivity a theorem proved for simply connected domains by G. Julia in the preliminaries to his prize memoir on the iteration of rational functions.*

Consider, in the plane of the variable z, a closed and bounded domain Δ, consisting of a two-dimensional continuum plus its boundary. Let a function $f(z)$ be analytic in Δ and assume throughout Δ values which correspond to inner points of Δ. Then $f(z)$ maps Δ conformally on a Riemann surface of one or more sheets, every interior and boundary point of which lies within Δ. If we project the points of this Riemann surface upon the z plane, we obtain a closed domain Δ_1. To every value assumed by $f(z)$ in Δ, no matter how many times, there corresponds one and only one point of Δ_1. The domain Δ_1 consists of inner points and boundary points, each boundary point being a limit point of inner points. Every value which $f(z)$ takes at an inner point of Δ gives an inner point of Δ_1; the boundary points of Δ_1 correspond only to values which $f(z)$ takes at boundary points, but not at inner points, of Δ; in certain cases, however, values which $f(z)$ takes only on the boundary of Δ may give inner points of Δ_1. It is easy to show that the inner points of Δ_1 form a continuum, but we shall not have occasion to use this fact.

It is evident that $f(z)$ transforms Δ_1 into a closed domain Δ_2 whose points are all inner points of Δ_1, Δ_2 into a smaller domain Δ_3, etc.

The theorem we are to prove states that:

(*a*) *In the transformation of Δ into Δ_1, one and only one point of Δ stays fixed.*

(*b*) *At this fixed point a, we have $|f'(a)| < 1$.*

(*c*) *The domains $\Delta, \Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$ converge to the fixed point.*

The application of this theorem to doubly connected regions shows that it is impossible to shrink a ring conformally into a ring situated in its interior; that is, if the given ring is bounded by certain curves C_1 and C_2 (C_2 interior to C_1), it is impossible to map the ring conformally in a one-to-one manner upon a ring bounded by curves Γ_1 and Γ_2, with Γ_1 interior to C_1 and C_2 interior to Γ_2. For it is clear that the points of the second ring would be mapped on a third ring, interior to the second, the third ring on a fourth in its interior, etc., and that the sequence of rings thus obtained could not converge to a point.

---

* G. Julia, Sur l'itération des fonctions rationnelles, Journ. de Math., 1918, p. 69.

It will be seen that Julia's proof, which consists in mapping Δ on a circle, in showing the existence of the fixed point by the theory of the zeros of analytic functions and in using Schwarz's lemma for the second and third items of the theorem, cannot be used when Δ is multiply connected. We shall handle the general case by means of Montel's normal families of functions, which Julia uses later himself, to the greatest advantage, in his remarkable paper.

2. We denote the nth iterate of $f(z)$ by $f_n(z)$. Since, for every z in Δ, $f(z)$ also lies in Δ, it is clear that all of the iterates

$$f_1(z),\ f_2(z),\ \cdots,\ f_n(z),\ \cdots \tag{1}$$

are defined throughout Δ, and have a common upper bound for their moduli in Δ; they constitute a normal family in the sense of Montel.

We can therefore select from the sequence (1) a new sequence

$$f_{i_1}(z),\ f_{i_2}(z),\ \cdots,\ f_{i_n}(z),\ \cdots \tag{2}$$

which converges uniformly in every closed domain interior to Δ to an analytic function $\varphi(z)$.*

The central part of the proof consists in showing that $\varphi(z)$ is a constant. Let us grant this fact for the moment; we shall see that the theorem follows directly from it.

The functions of the sequence (2) converge uniformly in Δ_1, which is a closed domain interior to Δ. The values assumed by $f_{i_n}(z)$ in Δ_1 are the affixes of the points of Δ_{i_n+1}. Hence if $\varphi(z)$ is a constant a, the domains

$$\Delta_{i_1+1},\ \Delta_{i_2+1},\ \cdots,\ \Delta_{i_n+1},\ \cdots$$

must converge to the point a. But since every Δ_n contains Δ_{n+1}, it is clear that the domains of the entire sequence

$$\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n,\ \cdots$$

converge to a.

The point a must stay fixed in the mapping, otherwise, for large values of n, the domain Δ_n which is very small, and close to a, could not contain Δ_{n+1}, which would be very small and close to $f(a)$. The fixed point a is clearly interior to every Δ_n.

Finally we must have $|f'(a)| < 1$. For since $f(a) = a$, we have

$$f_{i_n}'(a) = f'[f_{i_n-1}(a)]\cdot f'[f_{i_n-2}(a)]\cdots f'(a) = [f'(a)]^{i_n}.$$

---

* See, for instance, Montel, Les Séries de Polynomes, p. 22.

But

$$\varphi'(a) = \lim_{n \to \infty} f_{i_n}'(a),$$

and since $\varphi'(z)$ is identically zero, we must have

$$|f'(a)| < 1.$$

It remains now only to prove that $\varphi(z)$ is a constant.

We shall show first that if $\varphi(z)$ is not a constant, the points corresponding to the values assumed by $\varphi(z)$ in any one of the domains Δ_p $(p = 1, 2, \cdots)$,* are the points common to $\Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$.†

Firstly, if ω is interior to each of these domains, it is interior, in particular, to the domains of the sequence

$$\Delta_{i_1+p}, \Delta_{i_2+p}, \cdots, \Delta_{i_n+p}, \cdots.$$

Since $f_{i_n}(z)$ transforms Δ_p into Δ_{i_n+p}, each function of the sequence (2) assumes the value ω at some point of Δ_p. It is the simplest matter to show from this that $\varphi(z)$ also assumes the value ω in Δ_p.

Conversely, let $\varphi(z)$ assume the value ω at some point ζ of Δ_p. If $\varphi(z)$ is not a constant, there must exist a circumference lying wholly within Δ_{p-1}, with center at ζ, on no point of which $\varphi(z)$ is zero; there is a positive M such that, along this circumference,

$$|\varphi(z) - \omega| > M.$$

For n sufficiently great, we have, along the same circumference,

$$|f_{i_n}(z) - \varphi(z)| < M.$$

Hence, by a well-known theorem on the zeros of analytic functions, the function

$$f_{i_n}(z) - \omega = [\varphi(z) - \omega] + [f_{i_n}(z) - \varphi(z)]$$

has the same number of zeros within the circumference as $\varphi(z) - \omega$. What is the same, ω is interior to Δ_{i_n+p-1} for n sufficiently large, and hence it is interior to every Δ_n.

Thus $\varphi(z)$ transforms every Δ_p into the same domain δ which consists of the points common to all of the domains $\Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$.

In transforming Δ_p into δ, the boundary points of δ come only from the boundary points of Δ_p; in transforming Δ_{p+1} into δ, the boundary points of δ must come from the boundary points of Δ_{p+1}. But since the boundary points of Δ_{p+1} are inner points of Δ_p, the boundary points of Δ_{p+1} can give only inner points of δ.

---

* We take p at least equal to 1, because we have not shown that the sequence (2) converges in a domain which includes Δ.

† The proof of this fact will be recognized as a familiar argument in the theory of normal families.

The assumption that $\varphi(z)$ is not a constant has forced a contradiction. The theorem is proved.

COLUMBIA UNIVERSITY.